

A DIAGONAL ON THE ASSOCIAHEDRA

SAMSON SANEBLIDZE AND RONALD UMBLE¹

ABSTRACT. Let $C_*(K)$ denote the cellular chains on the Stasheff associahedra. We construct an explicit combinatorial diagonal $\Delta : C_*(K) \rightarrow C_*(K) \otimes C_*(K)$; consequently, we obtain an explicit diagonal on the A_∞ -operad. We apply the diagonal Δ to define the tensor product of A_∞ -(co)algebras in maximal generality.

1. INTRODUCTION

Let $C_*(K)$ denote the cellular chains on the disjoint union of the Stasheff associahedra $\{K_n\}_{n \geq 2}$. In this paper we construct an explicit combinatorial diagonal $\Delta : C_*(K) \rightarrow C_*(K) \otimes C_*(K)$ based on a direct decomposition of the top dimensional cells of K . This leads to an explicit diagonal on the A_∞ -operad and solves a long-standing problem. We apply the diagonal Δ to define the tensor product of A_∞ -(co)algebras in maximal generality. We also include an appendix in which we define an associahedral set \mathcal{K} and lift Δ to a diagonal on the chain complex of \mathcal{K} .

We mention that Chapoton [1], [2] constructed a diagonal on $C_*(K)$ of the form $\Delta : C_*(K_n) \rightarrow \bigoplus_{i+j=n} C_*(K_i) \otimes C_*(K_j)$, which coincides with the diagonal of Loday and Ronco [4] in dimension zero. Whereas Chapoton's diagonal is primitive on generators, our diagonal is defined by geometrically decomposing the generators. Thus the two are totally different.

2. THE STASHEFF ASSOCIAHEDRA

In his seminal papers of 1963, J. Stasheff [9] constructs the associahedra $\{K_{n+2}\}_{n \geq 0}$ as follows: Let $K_2 = *$; if K_{n+1} has been constructed, let

$$L_{n+2} = \bigcup_{\substack{r+s=n+3 \\ 1 \leq k \leq n-s+3}} (K_r \times K_s)_k$$

and define $K_{n+2} = CL_{n+2}$, i.e., the cone on L_{n+2} . The associahedron K_{n+2} is an n -dimensional polyhedron, which serves as a parameter space for homotopy associativity in $n+2$ variables. The top dimensional face of K_{n+2} corresponds to a pair of level 1 parentheses enclosing all $n+2$ indeterminants; each component $(K_{n-\ell+2} \times K_{\ell+1})_{i+1}$ of ∂K_{n+2} corresponds to a pair of level 2 parentheses enclosing $\ell+1$ indeterminants beginning with the $(i+1)^{st}$. We denote this parenthesization by

$$d_{(i,\ell)} = (x_1 \cdots (x_{i+1} \cdots x_{i+\ell+1}) \cdots x_{n+2})$$

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and refer to the inner and outer parentheses as the *first* and *last* pair, respectively. Note that indices i and ℓ are constrained by

$$\left\{ \begin{array}{ll} 0 \leq i \leq n \\ 1 \leq \ell \leq n, & i = 0 \\ 1 \leq \ell \leq n+1-i, & 1 \leq i \leq n \end{array} \right\}.$$

Thus, there is a one-to-one correspondence between $(n-1)$ -faces of K_{n+2} and parenthesizations $d_{(i,\ell)}$ of $n+2$ indeterminants.

Alternatively, K_{n+2} can be realized as a subdivision of the standard n -cube I^n in the following way: Let $\epsilon = 0, 1$. Label the endpoints of $K_3 = [0, 1]$ via $\epsilon \leftrightarrow d_{(\epsilon,1)}$. For $1 \leq i \leq n$, let $e_{i,\epsilon}^{n-1}$ denote the $(n-1)$ -face $(x_1, \dots, x_{i-1}, \epsilon, x_{i+1}, \dots, x_n) \subset I^n$ and obtain K_4 from $K_3 \times I = I^2$ by subdividing the edge $e_{1,1}^1$ as the union of intervals $1 \times I_{0,1} \cup 1 \times I_{1,\infty}$. Label the edges of K_4 as follows: $e_{1,0}^1 \leftrightarrow d_{(0,i)}$; $e_{2,1}^1 \leftrightarrow d_{(2,1)}$; $1 \times I_{0,1} \leftrightarrow d_{(1,1)}$; and $1 \times I_{1,\infty} \leftrightarrow d_{(1,2)}$ (see Figure 1). Now for $0 \leq i \leq j \leq \infty$, let $I_{i,j}$ denote the subinterval $[(2^i - 1)/2^i, (2^j - 1)/2^j] \subset I$, where $(2^\infty - 1)/2^\infty$ is defined to be 1. For $n > 2$, assume that K_{n+1} has been constructed and obtain K_{n+2} from $K_{n+1} \times I \approx I^n$ by subdividing the $(n-1)$ -faces $d_{(i,n-i)} \times I$ as unions $d_{(i,n-i)} \times I_{0,i} \cup d_{(i,n-i)} \times I_{i,\infty}$, $0 < i < n$. Label the $(n-1)$ -faces of K_{n+2} as follows:

Face of K_{n+2}	Label
$e_{\ell,0}^{n-1}$	$d_{(0,\ell)}, \quad 1 \leq \ell \leq n$
$e_{n,1}^{n-1}$	$d_{(n,1)},$
$d_{(i,\ell)} \times I$	$d_{(i,\ell)}, \quad 1 \leq \ell < n-i, \quad 0 < i < n-1$
$d_{(i,n-i)} \times I_{0,i}$	$d_{(i,n-i)}, \quad 0 < i < n$
$d_{(i,n-i)} \times I_{i,\infty}$	$d_{(i,n-i+1)}, \quad 0 < i < n$

In Figure 2 we have labeled the 2-faces of K_5 that are visible from the viewpoint of the diagram.

Compositions $d_{(i_m,\ell_m)} \cdots d_{(i_2,\ell_2)} d_{(i_1,\ell_1)}$ denote a successive insertion of $m+1$ pairs of parentheses into $n+2$ indeterminants as follows: Given $d_{(i_r,\ell_r)} \cdots d_{(i_1,\ell_1)}$, $1 \leq r < m$, regard each pair of level 2 parentheses and its contents as a single indeterminant and apply $d_{(i_{r+1},\ell_{r+1})}$. Conclude by inserting a last pair enclosing everything. Note that each parenthesization can be expressed as a unique composition $d_{(i_m,\ell_m)} \cdots d_{(i_1,\ell_1)}$ with $i_{r+1} \leq i_r$ for $1 \leq r < m$, in which case the parentheses inserted by $d_{(i_{r+1},\ell_{r+1})}$ begin at or to the left of the pair inserted by $d_{(i_r,\ell_r)}$. Such compositions are said to have *first fundamental form*. Thus for $0 \leq k < n$, the k -faces of K_{n+2} lie in one-to-one correspondence with compositions $d_{(i_{n-k},\ell_{n-k})} \cdots d_{(i_1,\ell_1)}$ in first fundamental form. The two extremes with m pairs of parentheses inserted as far to the left and right as possible, are respectively denoted by

$$d_{(0,\ell_m)} \cdots d_{(0,\ell_1)} \quad \text{and} \quad d_{(i_m,i_{m-1}-i_m)} \cdots d_{(i_1,i_0-i_1)},$$

where $i_0 = n+1$ and $i_{r+1} < i_r$, $0 \leq r < m$. In particular, the n -fold compositions

$$d_{(0,1)} \cdots d_{(0,1)} \quad \text{and} \quad d_{(1,1)} \cdots d_{(n,1)}$$

denote the extreme full parenthesizations of $n + 2$ indeterminants. When $m = 0$, define $d_{(i_m, \ell_m)} \cdots d_{(i_1, \ell_1)} = Id$.

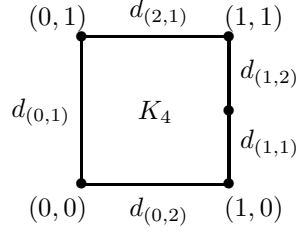


Figure 1: K_4 as a subdivision of $K_3 \times I$.

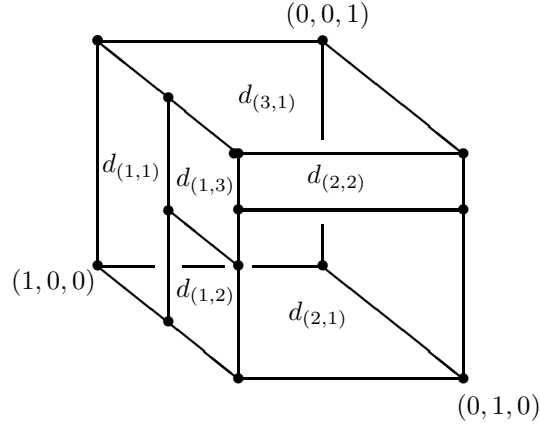


Figure 2: K_5 as a subdivision of $K_4 \times I$.

Alternatively, each face of K_{n+2} can be represented as a planar rooted tree (PRT) with $n + 2$ leaves; its leaves correspond to indeterminants and its nodes correspond to pairs of parentheses. Let T_{n+2} denote the PRT with $n + 2$ leaves attached to the root at a single node N_0 , called the *root node* (see Figure 3). The leaves correspond to a single pair of parentheses enclosing all $n + 2$ indeterminants. Now given an arbitrary PRT T , consider a node N of valence $r + 1 \geq 4$. Choose a neighborhood U of N that excludes the other nodes of T and note that $T_r \subseteq U \cap T$. Labeling from left to right, index the leaves of T_r from 1 to r as in Figure 3. Perform an (i, ℓ) -surgery at node N in the following way: Remove leaves $i + 1, \dots, i + \ell + 1$ of T_r , reattach them at a new node $N' \neq N$ and graft in a new branch connecting N to N' (see Figure 4). Now let $n \geq 1$. Given a parenthesization $d_{(i, \ell)}$ of $n + 2$ indeterminants, obtain the PRT $T_{n+2}^{(i, \ell)}$ from T_{n+2} by performing an (i, ℓ) -surgery at the root node N_0 as shown in Figure 4. Inductively, given a parenthesization $d_{(i_m, \ell_m)} \cdots d_{(i_1, \ell_1)}$ of $n + 2$ indeterminants expressed as a composition in first fundamental form, construct the corresponding PRT $T_{n+2}^{(i_1, \ell_1), \dots, (i_m, \ell_m)}$ as follows: Assume that $T_{n+2}^{(i_1, \ell_1), \dots, (i_r, \ell_r)}$ with nodes N_0, \dots, N_r has been constructed for some $1 \leq r < m$ and note that the root node N_0 has valence $n + 3 - \ell_1 - \dots - \ell_r$. Perform an (i_{r+1}, ℓ_{r+1}) -surgery at

N_0 and obtain $T_{n+2}^{(i_1, \ell_1), \dots, (i_{r+1}, \ell_{r+1})}$ containing a new node N_{r+1} and a new branch connecting N_0 to N_{r+1} . Finally, define $T_{n+2}^{(i_1, \ell_1), \dots, (i_m, \ell_m)} = T_{n+2}$ when $m = 0$ and obtain a one-to-one correspondence between k -faces of K_{n+2} , $0 \leq k \leq n$ and PRT's $T_{n+2}^{(i_1, \ell_1), \dots, (i_{n-k}, \ell_{n-k})}$ consisting of $n - k + 1$ nodes and $n + 2$ leaves. In particular, each vertex of K_{n+2} corresponds to a planar binary rooted tree $T_{n+2}^{(i_1, \ell_1), \dots, (i_n, \ell_n)}$ (see Figure 5).

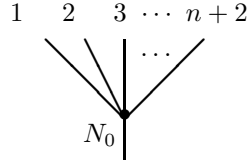
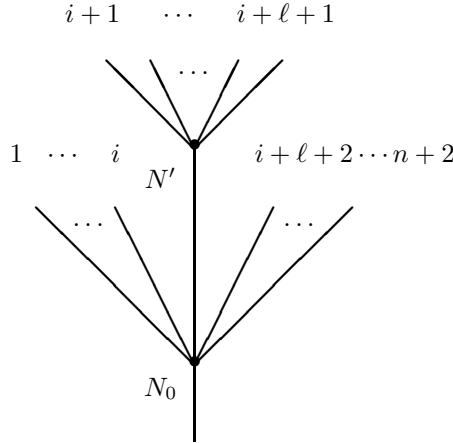
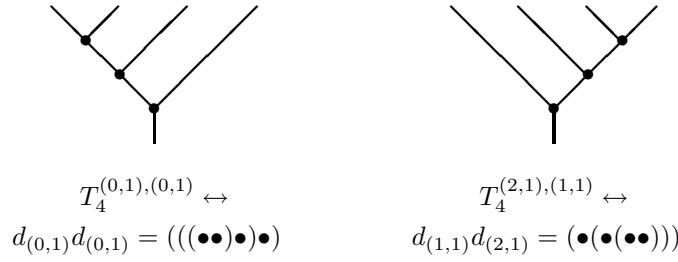
Figure 3: The corolla T_{n+2} Figure 4: The PRT $T_{n+2}^{(i, \ell)}$ 

Figure 5: Some binary PRT's.

Now given a k -face $a_k \subseteq K_{n+2}$, $k > 0$, consider the two vertices of a_k at which parentheses are shifted as far to the left and right as possible; we refer to these vertices as the minimal and maximal vertices of a_k , and denote them by a_k^{\min} and a_k^{\max} , respectively. In particular, the minimal and maximal vertices of K_{n+2} are

the origin and the vertex of I^n diagonally opposite to it, i.e.,

$$K_{n+2}^{\min} \leftrightarrow (0, 0, \dots, 0) \text{ and } K_{n+2}^{\max} \leftrightarrow (1, 1, \dots, 1);$$

the respective binary trees in Figure 5 correspond to K_4^{\min} and K_4^{\max} . Given a representation $T_{n+2}^{(i_1, \ell_1), \dots, (i_{n-k}, \ell_{n-k})}$ of a_k , construct the minimal (resp., maximal) tree of a_k by replacing each node of valence $r \geq 4$ with the planar binary rooted tree representing K_{r-1}^{\min} (resp., K_{r-1}^{\max}). Note that a_k^{\min} and a_k^{\max} determine a_k since their convex hull is a diagonal of a_k . But we can say more.

When a composition of face operators $d_{(i_m, \ell_m)} \cdots d_{(i_1, \ell_1)}$ is defined we refer to the sequence of lower indices $I = (i_1, \ell_1), \dots, (i_m, \ell_m)$ as an *admissible sequence of length m* ; if $d_{(i_m, \ell_m)} \cdots d_{(i_1, \ell_1)}$ has first fundamental form we refer to the sequence I as a *type I sequence of length m* . The set of all planar binary rooted trees

$$Y_{n+2} = \{T_{n+2}^I \mid I \text{ is a type I sequence of length } n\}$$

is a poset with partial ordering defined as follows (cf. [6]): Say that $T_{n+2}^{I_p} \leq T_{n+2}^{I_q}$ if there is an edge-path in K_{n+2} from vertex $T_{n+2}^{I_p}$ to vertex $T_{n+2}^{I_q}$ along which parentheses shift strictly to the right. This partial ordering can be expressed geometrically in terms of the following operation on trees: Let N_0 denote the root node of T_{n+2}^I and let N be a node joining some left branch L and a right branch or leaf R in T_{n+2}^I . Let N_L denote the node on L immediately above N . A *right-shift through node N* repositions N_L either at the midpoint of leaf R or midway between N and the node immediately above it. Then $T_{n+2}^{I_p} \leq T_{n+2}^{I_q}$ if there is a *right-shift sequence* of planar binary rooted trees $\{T_{n+2}^{I_r}\}_{p \leq r \leq q}$, i.e., for each $r < q$, tree $T_{n+2}^{I_{r+1}}$ is obtained from $T_{n+2}^{I_r}$ by a right-shift through some node in $T_{n+2}^{I_r}$ (see Figure 6).

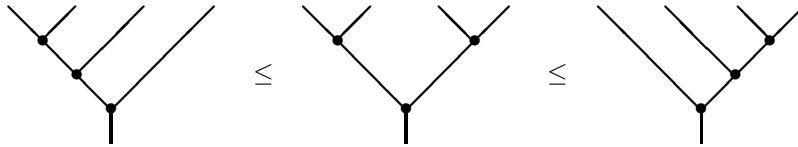


Figure 6: A right-shift sequence of planar binary trees

Let $I' = (i'_1, \ell'_1), \dots, (i'_m, \ell'_m)$ be an admissible sequence of length $m > 0$ and consider a node N' distinct from the root node N_0 in the PRT $T_{n+2}^{I'}$. Let N denote the node immediately below N' and let NN' denote the branch from N to N' ; we refer to the quotient space $T_{n+2}^{I'} = T_{n+2}^{I'}/NN'$ as the (N, N') -contraction of $T_{n+2}^{I'}$. Now given a type I sequence J' of length n , consider the planar binary tree $T_{n+2}^{J'}$ and let T_{n+2}^J be the PRT obtained from $T_{n+2}^{J'}$ by some sequence of k successive (N, N') -contractions. The subposet $Y_{n+2}^J \subseteq Y_{n+2}$ of all planar binary rooted trees from which T_{n+2}^J can be so obtained is exactly the poset of vertices of the k -face $a_k \subseteq K_{n+2}$ represented by T_{n+2}^J . In this way, we may regard a_k as the geometric realization of Y_{n+2}^J just as we regard a k -face of the standard n -simplex as the geometric realization of a $(k+1)$ -subset of a linearly ordered $(n+1)$ -set. In particular, K_{n+2} is the geometric realization of Y_{n+2} .

We summarize the discussion above as a proposition:

Proposition 1. *For $0 \leq k \leq n$, the following correspondences preserve combinatorial structure:*

$$\begin{aligned} \{k\text{-faces of } K_{n+2}\} &\leftrightarrow \left\{ \begin{array}{l} (n-k)\text{-fold compositions of face} \\ \text{operators in first fundamental form} \end{array} \right\} \\ &\leftrightarrow \left\{ \begin{array}{l} \text{Planar rooted trees with} \\ n-k+1 \text{ nodes and } n+2 \text{ leaves} \end{array} \right\} \\ &\leftrightarrow \left\{ \begin{array}{l} \text{Subposets of planar binary rooted trees } Y_{n+2}^J \\ \text{where } J \text{ is a type I sequence of length } n-k \end{array} \right\}. \end{aligned}$$

3. A DIAGONAL Δ ON $C_*(K_n)$

For notational simplicity, we suppress upper indices q_2, \dots, q_m in a composition $d_{(i_m, \ell_m)}^{q_m} \cdots d_{(i_2, \ell_2)}^{q_2} d_{(i_1, \ell_1)}^{q_1}$ when $q_{j+1} = q_j + 1$ for all $j \geq 1$; if, in addition, $q_1 = 1$, we suppress all q_i 's.

Definition 1. *Let $m \geq 2$. A sequence of lower indices $I = (i_1, \ell_1), \dots, (i_m, \ell_m)$ is admissible whenever the composition of face operators $d_{(i_m, \ell_m)}^{q_m} \cdots d_{(i_1, \ell_1)}^{q_1}$ is defined. The sequence I is a type I (resp. type II) sequence if I is admissible and $i_k \geq i_{k+1}$ (resp. $i_k \leq i_{k+1} + \ell_{k+1}$) for $1 \leq k < m$. The empty sequence ($m = 0$) and sequences of length 1 ($m = 1$) are sequences of types I and II. A composition of face operators $d_{(i_m, \ell_m)} \cdots d_{(i_2, \ell_2)} d_{(i_1, \ell_1)}^s$ has first (resp. second) fundamental form if $(i_1, \ell_1), \dots, (i_m, \ell_m)$ is a type I (resp. type II) sequence. When $m = 0$, the composition $d_{(i_m, \ell_m)} \cdots d_{(i_2, \ell_2)} d_{(i_1, \ell_1)}^s$ is defined to be the identity. An element $b = d_{(i_m, \ell_m)}^{q_m} \cdots d_{(i_2, \ell_2)}^{q_2} d_{(i_1, \ell_1)}^{q_1}(a)$ is expressed in first (resp. second) fundamental form as a face of a if*

$$d^{q_m} \cdots d^{q_2} d^{q_1} = (d \cdots d d^{s_m}) \cdots (d \cdots d d^{s_2}) (d \cdots d d^{s_1}),$$

where $s_1 < s_2 < \cdots < s_m$ and each composition $d_{(i_{s_j}, \ell_{s_j})} \cdots d_{(i_2, \ell_2)} d_{(i_1, \ell_1)}^{s_j}$ has first (resp. second) fundamental form.

Face operators satisfy the following relations:

$$d_{(i_p, \ell_p)}^p d_{(i_q, \ell_q)}^q = d_{(i_q, \ell_q)}^{q+1} d_{(i_p, \ell_p)}^p, \quad p < q \quad (1)$$

$$d_{(i_{q+1}, \ell_{q+1})}^{q+1} d_{(i_q, \ell_q)}^q = d_{(i_q - i_{q+1}, \ell_q)}^q d_{(i_{q+1}, \ell_{q+1} + \ell_q)}^q, \quad i_{q+1} \leq i_q \leq i_{q+1} + \ell_{q+1} \quad (2)$$

$$d_{(i_{q+1}, \ell_{q+1})}^{q+1} d_{(i_q, \ell_q)}^q = d_{(i_q, \ell_q)}^{q+1} d_{(i_{q+1} + \ell_q, \ell_{q+1})}^q, \quad i_q < i_{q+1}. \quad (3)$$

Furthermore, every composition of face operators can be uniquely transformed into first or second fundamental form by successive applications of face relations (1) to (3). For example, when $n = 2$, the following five face operators relate $T_4 \in K_4^2$ to the edges of the pentagon K_4 :

$$\begin{aligned} d_{(0,2)}(T_4) &\longmapsto ((\bullet\bullet\bullet)\bullet) && \in K_3 \times K_2 \\ d_{(1,2)}(T_4) &\longmapsto (\bullet(\bullet\bullet\bullet)) && \in K_3 \times K_2 \\ d_{(0,1)}(T_4) &\longmapsto ((\bullet\bullet)\bullet\bullet) && \in K_2 \times K_3 \\ d_{(1,1)}(T_4) &\longmapsto (\bullet(\bullet\bullet)\bullet) && \in K_2 \times K_3 \\ d_{(2,1)}(T_4) &\longmapsto (\bullet\bullet(\bullet\bullet)) && \in K_2 \times K_3. \end{aligned}$$

There are four compositions of face operators

$$d_{(i_2,1)}^1 d_{(i_1,2)}^1 : K_4 \rightarrow K_3 \times K_2 \rightarrow K_2 \times K_2 \times K_2$$

with $0 \leq i_1, i_2 \leq 1$, and six compositions

$$d_{(i_2,1)}^2 d_{(i_1,1)}^1 : K_4 \rightarrow K_2 \times K_3 \rightarrow K_2 \times K_2 \times K_2$$

with $0 \leq i_1 \leq 2$ and $0 \leq i_2 \leq 1$, which pair off via relations (1) to (3) and relate T_4 to each of the five vertices of K_4 :

$$\begin{aligned} d_{(0,1)}^2 d_{(0,1)}^1(T_4) &= d_{(0,1)}^1 d_{(0,2)}^1(T_4) \mapsto (((\bullet\bullet)\bullet)\bullet) \\ d_{(0,1)}^2 d_{(1,1)}^1(T_4) &= d_{(1,1)}^1 d_{(0,2)}^1(T_4) \mapsto ((\bullet(\bullet\bullet))\bullet) \\ d_{(1,1)}^2 d_{(1,1)}^1(T_4) &= d_{(0,1)}^1 d_{(1,2)}^1(T_4) \mapsto (\bullet((\bullet\bullet)\bullet)) \\ d_{(1,1)}^2 d_{(2,1)}^1(T_4) &= d_{(1,1)}^1 d_{(1,2)}^1(T_4) \mapsto (\bullet(\bullet(\bullet\bullet))) \\ d_{(0,1)}^2 d_{(2,1)}^1(T_4) &= d_{(1,1)}^2 d_{(0,1)}^1(T_4) \mapsto ((\bullet\bullet)(\bullet\bullet)). \end{aligned}$$

These relations encode the fact that when inserting two pairs of parentheses into a string of four variables either pair may be inserted first.

Given a type I sequence $(j_1, n_1 + 1), \dots, (j_k, n_k + 1)$ and a sequence (i_1, \dots, i_{k+1}) with $0 \leq i_q \leq n_q$, $1 \leq q \leq k + 1$, define a function of two variables

$$(3.1) \quad j(q, r) = \begin{cases} i_q + j_q + n_{q-1} + \dots + n_r + q - r, & 1 \leq r < q \\ i_q + j_q, & 1 \leq r = q \end{cases}$$

for $1 \leq r \leq q \leq k + 1$ (assuming $j_{k+1} = 0$), and let

$$(3.2) \quad \beta = \begin{cases} 1, & q = 1, \\ \max_{1 \leq r \leq q} \{r \mid j(q, r) \leq j_{r-1}\}, & q > 1, \end{cases}$$

(assuming $j_0 = \infty$.)

Definition 2. For $k \geq 0$, let $I_k = (j_1, n_1 + 1), (j_2, n_2 + 1), \dots, (j_k, n_k + 1)$ be a type I sequence. If $1 \leq q \leq k + 1$, the sign of the face $b = d_{(i_q, \ell_q)}^q(T_{n+2}^{I_k}) \in C_{n-k-1}(K_{n+2})$ is defined to be $(-1)^{\epsilon_1 + \epsilon_2}$, where

$$(3.3) \quad \begin{aligned} \epsilon_1 &= (i_q + 1)\ell_q + n_1 + \dots + n_{q-1}, \\ \epsilon_2 &= \begin{cases} 0, & 1 \leq \beta = q, \\ (\ell_q - 1)(n_{q-1} + \dots + n_\beta), & 1 \leq \beta < q, \end{cases} \end{aligned}$$

and β is defined by (3.2).

Let us construct an explicit diagonal on associahedra in terms of compositions of face operators in first and second fundamental form; the formulas we obtain also determine a DG coalgebra structure on $(C_*(K_{n+2}), d)$.

We begin with an overview of the geometric ideas involved. Let $0 \leq q \leq n$ and let I_{n-q} be a type I sequence. The q -dimensional generator $a_q = T_{n+2}^{I_{n-q}}$ is associated with a face of K_{n+2} corresponding to $n + 2$ indeterminants with $n - q + 1$ pairs of parentheses. Identify a_q with its associated face of K_{n+2} and consider the minimal and maximal vertices a_q^{\min} and a_q^{\max} of a_q . Define the primitive terms of ΔT_{n+2} to be

$$T_{n+2}^{\min} \otimes T_{n+2} + T_{n+2} \otimes T_{n+2}^{\max}.$$

Let $0 < p < p + q = n$ and consider distinct p -faces b and b' of T_{n+2} . Say that $b \leq b'$ if there is a path of p -faces from b to b' along which parentheses shift strictly to the right. Now given a q -dimensional face a_q of T_{n+2} such that $a_q^{\min} \neq T_{n+2}^{\min}$, there is a unique path of p -faces $b_1 \leq b_2 \leq \dots \leq b_r$ with minimal length such that

$T_{n+2}^{\min} = b_1^{\min}$ and $a_q^{\min} = b_r^{\max}$. Up to sign, we define the non-primitive terms of ΔT_{n+2} to be

$$\sum \pm b_j \otimes a_q.$$

To visualize this, consider the edge $d_{(1,2)}(T_4) \in C_1(K_4)$ whose minimal vertex is the point $(1, \frac{1}{2})$ (see Figure 7). The edges $d_{(0,2)}(T_4) \leq d_{(1,1)}(T_4)$ form a path of minimal length from $(0,0)$ to $(1, \frac{1}{2})$. Consequently, ΔT_4 contains the non-primitive terms $\{(\pm d_{(0,2)} \pm d_{(1,1)}) \otimes d_{(1,2)}\}(T_4 \otimes T_4)$.

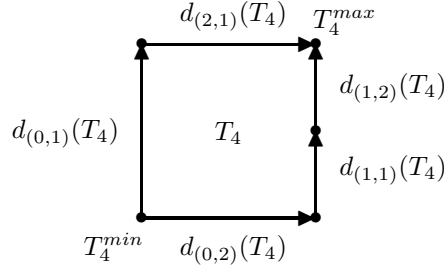


Figure 7: Edge paths from T_4^{\min} to T_4^{\max} .

Precisely, for $T_2 \in C_*(K_2)$ define $\Delta T_2 = T_2 \otimes T_2$; inductively, assume that the map $\Delta : C_*(K_{i+2}) \rightarrow C_*(K_{i+2}) \otimes C_*(K_{i+2})$ has been defined for all $i < n$. For $T_{n+2} \in C_n(K_{n+2})$ define

$$\Delta T_{n+2} = \sum_{0 \leq p \leq p+q=n} (-1)^\epsilon d_{(i'_q, \ell'_q)} \cdots d_{(i'_1, \ell'_1)}(T_{n+2}) \otimes d_{(i_p, \ell_p)} \cdots d_{(i_1, \ell_1)}(T_{n+2})$$

where

$$\epsilon = \sum_{j=1}^q i'_j(\ell'_j + 1) + \sum_{k=1}^p (i_k + k + p + 1)\ell_k,$$

and lower indices $((i_1, \ell_1), \dots, (i_p, \ell_p); (i'_1, \ell'_1), \dots, (i'_q, \ell'_q))$ range over all solutions of the following system of inequalities:

$$(3.4) \quad \left\{ \begin{array}{ll} 1 \leq i_j < i_{j-1} \leq n+1 & (1) \\ 1 \leq \ell_j \leq n+1 - i_j - \ell_{(j-1)} & (2) \\ 0 \leq i'_k \leq \min_{o'(t_k) < r < k} \{i'_r, i_{t_k} - \ell'_{(o'(t_k))}\} & (3) \\ 1 \leq \ell'_k = \epsilon_k - i'_k - \ell'_{(k-1)} & (4) \end{array} \right\}_{\substack{1 \leq j \leq p \\ 1 \leq k \leq q}},$$

where

$$\begin{aligned} \{\epsilon_1 < \cdots < \epsilon_q\} &= \{1, \dots, n\} \setminus \{i_1, \dots, i_p\}; \\ \epsilon_0 &= \ell_0 = \ell'_0 = i_{p+1} = i'_{q+1} = 0; \\ i_0 &= i'_0 = \epsilon_{q+1} = \ell_{(p+1)} = \ell'_{(q+1)} = n+1; \\ \ell_{(u)} &= \sum_{j=0}^u \ell_j \text{ for } 0 \leq u \leq p+1; \end{aligned}$$

$$\ell'_{(u)} = \sum_{k=0}^u \ell'_k \text{ for } 0 \leq u \leq q+1;$$

$$(3.5) \quad \begin{aligned} t_u &= \min \{r \mid i_r + \ell_{(r)} - \ell_{(o(u))} > \epsilon_u > i_r\}; \\ o(u) &= \max \{r \mid i_r \geq \epsilon_u\} \text{ and } o'(u) = \max \{r \mid \epsilon_r \leq i_u\}. \end{aligned}$$

Extend Δ multiplicatively to all of $C_*(K_{n+2})$, using the fact that the cells of K_{n+2} are products of cells K_{i+2} with $i < n$.

Note that right-hand and left-hand factors in each component of ΔT_{n+2} are expressed in first and second fundamental form, respectively. In particular, the terms given by the extremes $p = 0$ and $p = n$ are the primitive terms of ΔT_{n+2} :

$$\{d_{(0,1)} \cdots d_{(0,1)} \otimes 1 + 1 \otimes d_{(1,1)} \cdots d_{(n,1)}\} (T_{n+2} \otimes T_{n+2}).$$

The sign in $\Delta T_{n+2} = \sum (-1)^\epsilon b \otimes a$ is the product of five signs: $(-1)^\epsilon = \text{sgn}(b) \cdot \text{sgn}(b_0) \cdot \text{sgn}(b_0, a_1) \cdot \text{sgn}(a_1) \cdot \text{sgn}(a)$, where the face b_0 is obtained from b by using the same ϵ 's but all $i'_k = 0$ (i.e., ℓ'_k is replaced by $\ell'_k + i'_k - i'_{k-1}$), the face a_1 is obtained from a by using the same i_r 's but all $\ell_r = 1$ and $\text{sgn}(b_0, a_1)$ is the sign of the unshuffle $\{i_p < \cdots < i_1, \epsilon_1 < \epsilon_2 < \cdots < \epsilon_q\}$. Geometrically, b_0 and a_1 lie on orthogonal faces of the cube I^n and are uniquely defined by the property that the canonical cellular projection $K_{n+2} \rightarrow I^n$ maps $b_0 \mapsto (x_1, \dots, x_{\epsilon_1-1}, 0, \dots, x_{\epsilon_q-1}, 0, \dots, x_n)$ and $a_1 \mapsto (x_1, \dots, x_{i_p-1}, 1, \dots, x_{i_1-1}, 1, \dots, x_n)$.

Example 1. We obtain by direct calculation:

For $T_3 \in C_1(K_3)$:

$$\Delta T_3 = (d_{(0,1)} \otimes 1 + 1 \otimes d_{(1,1)})(T_3 \otimes T_3).$$

For $T_4 \in C_2(K_4)$:

$$\begin{aligned} \Delta T_4 &= (d_{(0,1)} d_{(0,1)} \otimes 1 + 1 \otimes d_{(1,1)} d_{(2,1)} + d_{(0,2)} \otimes d_{(1,1)} \\ &\quad + d_{(0,2)} \otimes d_{(1,2)} + d_{(1,1)} \otimes d_{(1,2)} - d_{(0,1)} \otimes d_{(2,1)})(T_4 \otimes T_4). \end{aligned}$$

For $T_5 \in C_3(K_5)$:

$$\begin{aligned} \Delta T_5 &= (d_{(0,1)} d_{(0,1)} d_{(0,1)} \otimes 1 + 1 \otimes d_{(1,1)} d_{(2,1)} d_{(3,1)} + \\ &\quad + d_{(0,3)} \otimes d_{(1,1)} d_{(2,2)} - d_{(1,1)} \otimes d_{(1,2)} d_{(3,1)} + d_{(0,1)} d_{(0,2)} \otimes d_{(1,2)} \\ &\quad + d_{(0,3)} \otimes d_{(1,2)} d_{(2,1)} - d_{(1,2)} \otimes d_{(1,2)} d_{(2,1)} + d_{(0,1)} d_{(0,2)} \otimes d_{(1,3)} \\ &\quad + d_{(0,3)} \otimes d_{(1,1)} d_{(2,1)} - d_{(1,2)} \otimes d_{(1,1)} d_{(2,2)} - d_{(0,2)} d_{(0,1)} \otimes d_{(2,1)} \\ &\quad + d_{(0,1)} \otimes d_{(2,1)} d_{(3,1)} + d_{(0,1)} d_{(0,1)} \otimes d_{(3,1)} + d_{(0,2)} d_{(0,1)} \otimes d_{(2,2)} \\ &\quad + d_{(2,1)} \otimes d_{(1,1)} d_{(2,2)} + d_{(1,1)} d_{(0,1)} \otimes d_{(2,2)} + d_{(0,2)} d_{(1,1)} \otimes d_{(1,2)} \\ &\quad - d_{(0,2)} \otimes d_{(1,1)} d_{(3,1)} + d_{(1,1)} d_{(1,1)} \otimes d_{(1,3)} + d_{(0,2)} d_{(1,1)} \otimes d_{(1,3)} \\ &\quad - d_{(0,2)} \otimes d_{(1,2)} d_{(3,1)} + d_{(0,1)} d_{(0,2)} \otimes d_{(1,1)})(T_5 \otimes T_5). \end{aligned}$$

Our main result, stated as the following theorem, is proved in Section 4:

Theorem 1. For each $n \geq 0$, the map $\Delta : C_*(K_{n+2}) \rightarrow C_*(K_{n+2}) \otimes C_*(K_{n+2})$ defined above is a chain map.

Identify the sequence of cellular chain complexes $\{C_*(K_n)\}_{n \geq 2}$ with the A_∞ -operad \mathcal{A}_∞ [5]. Since Δ is extended multiplicatively on decomposable faces, we immediately obtain:

Corollary 1. The sequence of chain maps $\{\Delta : C_*(K_n) \rightarrow C_*(K_n) \otimes C_*(K_n)\}_{n \geq 2}$ induces a morphism of operads $\mathcal{A}_\infty \rightarrow \mathcal{A}_\infty \otimes \mathcal{A}_\infty$.

4. A PROOF OF THEOREM 1

In this section we prove that the diagonal Δ defined in Section 3 is a chain map. We begin with some preliminaries. It will be convenient to rewrite face relation (3) as follows:

$$d_{(i_{q+1}, \ell_{q+1})}^{q+1} d_{(i_q, \ell_q)}^q = d_{(i_q - \ell_{q+1}, \ell_q)}^{q+1} d_{(i_{q+1}, \ell_{q+1})}^q; \quad i_q > i_{q+1} + \ell_{q+1}. \quad (3')$$

Definition 3. For $1 \leq k \leq m$, the k^{th} left-transfer τ_ℓ^k of a composition $d_{(i_m, \ell_m)} \cdots d_{(i_2, \ell_2)} d_{(i_1, \ell_1)}$ in first fundamental form is one of the following compositions:

- (a) If $i_{k+1} + \ell_{k+1} \geq i_k$, apply face relation (2) to $d^{k+1} d^k$ and obtain

$$\cdots d_{(i_k - i_{k+1}, \ell_k)}^k d_{(i_{k+1}, \ell_{k+1} + \ell_k)}^k \cdots;$$

then successively apply face relation (1) to $d^{j+2} d^j$ for $j = k, k+1, \dots, m-2$, and obtain

$$\tau_\ell^k = d_{(i_k - i_{k+1}, \ell_k)}^k d_{(i_m, \ell_m)}^{m-1} \cdots d_{(i_{k+2}, \ell_{k+2})}^{k+1} d_{(i_{k+1}, \ell_{k+1} + \ell_k)}^k \cdots d_{(i_1, \ell_1)}^1.$$

- (b) If $i_{k+1} + \ell_{k+1} < i_k \leq i_{q+1} + \ell_{(q+1)} - \ell_{(k)}$ for some smallest integer $q > k$, successively apply face relation (3') to $d^{j+1} d^j$ for $j = k, k+1, \dots, q$, and obtain

$$\cdots d_{(i_{q+2}, \ell_{q+2})}^{q+2} d_{(i_k + \ell_{(k)} - \ell_{(q+1)}, \ell_k)}^{q+1} d_{(i_{q+1}, \ell_{q+1})}^q \cdots d_{(i_{k+1}, \ell_{k+1})}^k d_{(i_{k-1}, \ell_{k-1})} \cdots.$$

Apply face relation (2) to $d^{q+1} d^q$; then successively apply face relation (1) to $d^{j+2} d^j$ for $j = q, q+1, \dots, m-2$, and obtain

$$\tau_\ell^k = d_{(i, \ell_k)}^q d_{(i_m, \ell_m)}^{m-1} \cdots d_{(i_{k+1}, \ell_{k+1})}^k d_{(i_{k-1}, \ell_{k-1})} \cdots d_{(i_1, \ell_1)},$$

where $i = i_k - i_{q+1} + \ell_{(k)} - \ell_{(q)}$.

- (c) Otherwise, successively apply face relation (3') to $d^{j+1} d^j$ for $j = k, k+1, \dots, m-1$, and obtain

$$\tau_\ell^k = d_{(i, \ell_k)}^m d_{(i_m, \ell_m)}^{m-1} \cdots d_{(i_{k+1}, \ell_{k+1})}^k d_{(i_{k-1}, \ell_{k-1})} \cdots d_{(i_1, \ell_1)}$$

where $i = i_k + \ell_{(k)} - \ell_{(m)}$.

Definition 4. For $1 \leq k \leq m$, the k^{th} right-transfer τ_r^k of a composition $d_{(i_m, \ell_m)} \cdots d_{(i_2, \ell_2)} d_{(i_1, \ell_1)}$ in first fundamental form is one of the following compositions:

- (a) If $i_p + \ell_{(p)} \leq i_k + \ell_{(k)} < i_{p-1} + \ell_{(p-1)}$ for some greatest integer $1 < p \leq k$, successively apply face relation (3') to $d^{j+1} d^j$ for $j = p-1, \dots, 2, 1$; $j = p, \dots, 3, 2$; \dots , $j = k-1, k-2, \dots, k-(p-1)$. Then apply face relation (2) to $d^{j+1} d^j$ for $j = k-p, \dots, 2, 1$ and obtain

$$\begin{aligned} \tau_r^k &= d_{(i_m, \ell_m)} \cdots d_{(i_{k+1}, \ell_{k+1})} d_{(i_{p-1} - \ell_{(p-1)}, \ell_{p-1})}^k \\ &\quad \cdots d_{(i_1 - \ell_{(1)}, \ell_1)}^{k-p+2} d_{(i_{k-1} - i_k, \ell_{k-1})}^{k-p} \cdots d_{(i_p - i_k, \ell_p)}^1 d_{(i_k, \ell)}^1, \end{aligned}$$

where $\ell = \ell_{(k)} - \ell_{(p-1)}$.

- (b) Otherwise, successively apply face relation (2) to $d^{j+1}d^j$ for $j = k-1, k-2, \dots, 2, 1$, and obtain

$$\tau_r^k = d_{(i_m, \ell_m)} \cdots d_{(i_{k+1}, \ell_{k+1})} d_{(i_{k-1}-i_k, \ell_{k-1})}^{k-1} \cdots d_{(i_1-i_k, \ell_1)}^1 d_{(i_k, \ell_{(k)})}^1.$$

Note that if $I = (i_1, \ell_1), \dots, (i_m, \ell_m)$ is a type I sequence and $a = d_I(T_{n+2}) \in C_{n-m}(K_{n+2})$, then for each k , the k^{th} right transfer $\tau_r^k(T_{n+2})$ expresses a in first fundamental form as a face of some $(n-1)$ -face b of T_{n+2} . The expressions $\tau_r^1(T_{n+2}), \dots, \tau_r^m(T_{n+2})$ determine the m distinct $(n-1)$ -faces b containing a .

There are analogous left and right transfers for compositions in second fundamental form.

Definition 5. For $1 \leq k \leq m$, the k^{th} left-transfer τ_ℓ^k of a composition $d_{(i_m, \ell_m)} \cdots d_{(i_2, \ell_2)} d_{(i_1, \ell_1)}$ in second fundamental form is one of the following compositions:

- (a) If $i_{k+1} \leq i_k$, apply face relation (2) to $d^{k+1}d^k$, then successively apply face relation (1) to $d^{j+2}d^j$ for $j = k, k+1, \dots, m-2$, and obtain

$$\tau_\ell^k = d_{(i_k-i_{k+1}, \ell_k)}^k d_{(i_m, \ell_m)}^{m-1} \cdots d_{(i_{k+2}, \ell_{k+2})}^{k+1} d_{(i_{k+1}, \ell_{k+1}+\ell_k)}^k \cdots d_{(i_1, \ell_1)}^1.$$

- (b) If $i_{k+1} > i_k \geq i_{q+1}$ for some smallest integer $q > k$, successively apply face relation (3) to $d^{j+1}d^j$ for $j = k, k+1, \dots, q$. Apply face relation (2) to $d^{q+1}d^q$; then successively apply face relation (1) to $d^{j+2}d^j$ for $j = q, q+1, \dots, m-2$, and obtain

$$\begin{aligned} \tau_\ell^k &= d_{(i_k-i_{q+1}, \ell_k)}^q d_{(i_m, \ell_m)}^{m-1} \cdots d_{(i_{q+2}, \ell_{q+2})}^{q+1} d_{(i_{q+1}, \ell_{q+1}+\ell_k)}^q d_{(i_q+\ell_k, \ell_q)}^{q-1} \\ &\quad \cdots d_{(i_{k+1}+\ell_k, \ell_{k+1})}^k d_{(i_{k-1}, \ell_{k-1})} \cdots d_{(i_1, \ell_1)}. \end{aligned}$$

- (c) Otherwise, successively apply face relation (3) to $d^{j+1}d^j$ for $j = k, k+1, \dots, m-1$, and obtain

$$\tau_\ell^k = d_{(i_k, \ell_k)}^m d_{(i_m+\ell_k, \ell_m)}^{m-1} \cdots d_{(i_{k+2}+\ell_k, \ell_{k+2})}^{k+1} d_{(i_{k+1}+\ell_k, \ell_{k+1})}^k d_{(i_{k-1}, \ell_{k-1})} \cdots d_{(i_1, \ell_1)}.$$

Definition 6. For $1 \leq k \leq m$, the k^{th} right-transfer τ_r^k of a composition $d_{(i_m, \ell_m)} \cdots d_{(i_2, \ell_2)} d_{(i_1, \ell_1)}$ in second fundamental form is one of the following compositions:

- (a) If $i_{p-1} < i_k \leq i_p$ for some greatest integer $1 < p \leq k$, successively apply face relation (3) to $d^{j+1}d^j$ for $j = p-1, \dots, 2, 1$; $j = p, \dots, 3, 2$; \dots ; $j = k-1, k-2, \dots, k-(p-1)$. Then apply face relation (2) to $d^{j+1}d^j$ for $j = k-p, \dots, 2, 1$ and obtain

$$\begin{aligned} \tau_r^k &= d_{(i_m, \ell_m)} \cdots d_{(i_{k+1}, \ell_{k+1})} d_{(i_{p-1}, \ell_{p-1})}^k \cdots d_{(i_1, \ell_1)}^{k-p+2} d_{(i_{k-1}-i_k, \ell_{k-1})}^{k-p} \\ &\quad \cdots d_{(i_p-i_k, \ell_p)}^1 d_{(i_k+\ell_{(p-1)}, \ell_{(k)}-\ell_{(p-1)})}^1. \end{aligned}$$

- (b) Otherwise, successively apply face relation (2) to $d^{j+1}d^j$ for $j = k-1, \dots, 2, 1$, and obtain:

$$\tau_r^k = d_{(i_m, \ell_m)} \cdots d_{(i_{k+1}, \ell_{k+1})} d_{(i_{k-1}-i_k, \ell_{k-1})}^k \cdots d_{(i_1-i_k, \ell_1)}^1 d_{(i_k, \ell_{(k)})}^1.$$

Once again, if $I = (i_1, \ell_1), \dots, (i_m, \ell_m)$ is a type II sequence and $a = d_I(T_{n+2}) \in C_{n-m}(K_{n+2})$, then for each k , the k^{th} right-transfer $\tau_r^k(T_{n+2})$ expresses a in second fundamental form as a face of some $(n-1)$ -face b ; the expressions $\tau_r^1(T_{n+2}), \dots, \tau_r^m(T_{n+2})$ determine the m distinct $(n-1)$ -faces b containing a . Thus two $(n-m)$ -faces a_1 and a_2 expressed in first and second fundamental form, respectively, are contained in the same $(n-1)$ -face b if and only if there exist right transfers $\tau_r^{k_j}$ such that $a_j = \tau_r^{k_j}(T_{n+2})$, $j = 1, 2$, and $b = d_{(i, \ell)}(T_{n+2})$, where $d_{(i, \ell)}$ is the right-most face operator common to $\tau_r^{k_1}$ and $\tau_r^{k_2}$. We state this formally in the following lemma:

Lemma 1. *Let $0 \leq m_1, m_2 \leq n$ and assume that $a_1 = d_{(i_{m_1}, \ell_{m_1})} \cdots d_{(i_1, \ell_1)}(T_{n+2})$ and $a_2 = d_{(i'_{m_2}, \ell'_{m_2})} \cdots d_{(i'_1, \ell'_1)}(T_{n+2})$ are expressed in first and second fundamental form, respectively. Then a_1 and a_2 are contained in the same $(n-1)$ -face $d_{(i, \ell)}(T_{n+2})$ if and only if there exist integers $k_j \leq m_j$ and greatest integers $1 \leq p_j < k_j$ for $j = 1, 2$, such that*

$$i'_{p_2} < i'_{k_2},$$

$$i_{k_1} + \ell_{(k_1)} - \ell_{(p_1)} < i_{p_1}$$

and

$$i = i_{k_1} = i'_{k_2} + \ell'_{(p_2)},$$

$$\ell = \ell_{(k_1)} - \ell_{(p_1)} = \ell'_{(k_2)} - \ell'_{(p_2)}.$$

Consider a solution $((i_1, \ell_1), \dots, (i_p, \ell_p); (i'_1, \ell'_1), \dots, (i'_q, \ell'_q))$ of system (3.4) and its related (p, q) -unshuffle $\{i_p < \dots < i_1, \epsilon_1 < \epsilon_2 < \dots < \epsilon_q\}$. Given $1 \leq k \leq q+1$, let $k_1 = k-1$, $k_{j+1} = o'(t_{k_j})$ for $j \geq 1$ and note that $k_{j+1} < k_j$ for all j . For $0 \leq m < \ell'_k$, consider the following *selection algorithm*:

```

if  $k = 1$  then  $z = p + 1 - i'_1 - m$ 
else  $z = n + 2$ ;  $j = 0$ 
  repeat
     $j \leftarrow j + 1$ 
    if  $i'_{k_j} < i'_k + m$  then  $i_z = \epsilon_{k_j} - i'_{k_j} + i'_k + m$ 
    if  $i_{t_{k_j}} - \ell'_{(k_{j+1})} = i'_k + m$  then  $z = t_{k_j}$ 
  until  $z < n + 2$ 
endif

```

It will be clear from the proof of Lemma 2 below that the selection algorithm eventually terminates.

Example 2. *Let $p = q = 4$ and consider the following solution of system (3.4):*

$$((7, 1), (6, 1), (4, 2), (2, 3); (0, 1), (1, 1), (1, 2), (0, 4)).$$

Then $t_1 = 5$, $t_2 = 4$, $t_3 = 3$, $t_4 = 4$ and the selection algorithm produces the following:

k	1	2	3	3	4	4	4	4
m	0	0	0	1	0	1	2	3
z	5	5	4	3	5	4	2	1

The key to our proof that Δ is a chain map is given by our next lemma.

Lemma 2. *Given $1 \leq k \leq q+1$ and $0 \leq m < \ell'_k$, let z be the integer given by the selection algorithm.*

- (a) $z > o(k)$
- (b) $i_z + \ell_{(z)} - \ell_{(o(k))} \geq \epsilon_k$.
- (c) $i'_k + m = i_z - \ell'_{(o'(z))}$;
- (d) If $o'(z) < r < k$, then $i'_k + m \leq i'_r$;
- (e) If $i'_k + m > \min_{o'(t_k) < r < k} \{i'_r, i_{t_k} - \ell'_{(o'(t_k))}\}$, then
 - (e.1) $z < t_k$ and $i_z + \ell_{(z)} - \ell_{(o(k))} = \epsilon_k$;
 - (e.2) $o(k) = \max_{r < z} \{r \mid i_r > i_z + \ell_{(z)} - \ell_{(r)}\}$;
 - (e.3) $o'(z) = \max_{r < k} \{r \mid i'_r < i'_k + m\}$.

Proof:

(a) When $k = 1$, $z = (p+1-\epsilon_1) + (\ell'_1 - m) > p+1-\epsilon_1 = o(1)$. For $k > 1$, first note that $i_{t_{k_j}} < \epsilon_{k_j} < \epsilon_k \leq i_{o(k)}$ for all $j \geq 1$, by the definition of t_{k_j} in (3.5). So the result follows whenever $i_z \leq i_{t_{k_j}}$ for some $j \geq 1$. If $i'_{k-1} < i'_k + m$, then $i_z = \epsilon_{k-1} + i'_k + m - i'_{k-1} < \epsilon_{k-1} + i'_k + \ell'_{k-1} - i'_{k-1} = \epsilon_k$. If $i'_{k_j} < i'_k + m$ for some $j \geq 2$ and $i'_k + m \leq i'_{k_r} \leq i_{t_{k_r}} - \ell'_{(k_{r+1})}$ for all $r < j$, where the later follows from inequality (3) of system (3.4), then $i_z = \epsilon_{k_j} + i'_k + m - i'_{k_j} < \epsilon_{k_j} + i_{t_{k_{j-1}}} - \ell'_{(k_j)} - i'_{k_j} = i_{t_{k_{j-1}}}$.

(b) When $k = 1$, indices $0 = i_{p+1} < i_p < \dots < i_{p+2-\epsilon_1}$ are consecutive; hence $i_r + \ell_{(r)} = i_{r-1} + \ell_{(r-1)} + \ell_r - 1 \geq i_{r-1} + \ell_{(r-1)}$ for $p+3-\epsilon_1 \leq r \leq p+1$. Since $z \geq p+2-\epsilon_1$ we have $i_z + \ell_{(z)} - \ell_{(o(1))} \geq i_{p+2-\epsilon_1} + \ell_{(p+2-\epsilon_1)} - \ell_{(p+1-\epsilon_1)} = \epsilon_1 - 1 + \ell_{p+2-\epsilon_1} \geq \epsilon_1$. So consider $k > 1$. If $i'_{k-1} < i'_k + m$, then $\epsilon_{k-1} < i_z < \epsilon_k < i_{o(k)}$ so that $\epsilon_k = i_z + z - o(k) \leq i_z + \ell_{(z)} - \ell_{(o(k))}$. If $i'_{k-1} \geq i'_k + m$ and $z = t_{k-1}$, then $o(k-1) - o(k) = \epsilon_k - \epsilon_{k-1} - 1$ so that $i_z + \ell_{(z)} - \ell_{(o(k))} = i_z + \ell_{(z)} - \ell_{(o(k-1))} + \ell_{(o(k-1))} - \ell_{(o(k))} > \epsilon_{k-1} + \epsilon_k - \epsilon_{k-1} - 1 = \epsilon_k - 1$ by the definition of t_{k-1} in (3.5). If $i'_{k_2} < i'_k + m$ then $\epsilon_{k_2} < i_z < i_{t_{k_1}} < \epsilon_{k_1}$ so that $\ell_{(z)} - \ell_{(t_{k_1})} \geq z - t_{k_1} = i_{t_{k_1}} - i_z$, where equality holds iff $\ell_{t_{k_1}+1} = \dots = \ell_z = 1$; hence $i_z + \ell_{(z)} - \ell_{(o(k))} = i_z + \ell_{(z)} - \ell_{(t_{k_1})} + \ell_{(t_{k_1})} - \ell_{(o(k))} \geq i_z + i_{t_{k_1}} - i_z + \ell_{(t_{k_1})} - \ell_{(o(k))} > \epsilon_k - 1$ by the previous calculation. If $i'_{k_2} \geq i'_k + m$ and $z = t_{k_2}$, then $i_z + \ell_{(z)} - \ell_{(o(k_1))} = i_z + \ell_{(z)} - \ell_{(o(k_2))} + \ell_{(o(k_2))} - \ell_{(o(k_1))} > \epsilon_{k_2} + \ell_{(o(k_2))} - \ell_{(o(k_1))} \geq i_{t_{k_1}} + \ell_{(t_{k_1})} - \ell_{(o(k_2))} + \ell_{(o(k_2))} - \ell_{(o(k_1))} > \epsilon_{k_1}$ since $t_{k_1} < t_{k_2}$ implies $i_{t_{k_1}} + \ell_{(t_{k_1})} - \ell_{(o(k_2))} \leq \epsilon_{k_2}$ by the definition of t_{k_2} . Thus $i_z + \ell_{(z)} - \ell_{(o(k))} = i_z + \ell_{(z)} - \ell_{(o(k_1))} + \ell_{(o(k_1))} - \ell_{(o(k))} > \epsilon_k - 1$. Note that in general, $i_{t_{k_j}} + \ell_{(t_{k_j})} - \ell_{(o(k_1))} > \epsilon_{k_1}$ so that $i_{t_{k_j}} + \ell_{(t_{k_j})} - \ell_{(o(k))} \geq \epsilon_k$ for all $j \geq 1$. So if necessary, continue in like manner until the desired z is found at which point the result follows.

(c) This result follows immediately from the choice of z .

(d) Note that $k > 1$. If $i'_{k_1} < i'_k + m$, then $o'(z) = k-1$ and the case is vacuous. So assume that $i'_k + m \leq i'_{k_1}$. If either $i'_k + m = i_{t_{k_1}} - \ell'_{(k_2)}$ or $i'_{k_2} < i'_k + m$, then $o'(z) = o'(t_{k_1})$ and $i'_{k_1} \leq i'_r$ for $o'(z) < r < k$. Otherwise, $i'_k + m \leq \min \{i'_{k_1}, i'_{k_2}\}$.

If either $i'_k + m = i_{t_{k_2}} - \ell'_{(k_3)}$ or $i'_{k_3} < i'_k + m$, then $o'(z) = o'(t_{k_2})$ and $i'_{k_2} \leq i'_r$ for $o'(z) < r \leq k_2$. But $i'_{k_1} \leq i'_r$ for $k_2 = o'(t_{k_1}) < r < k$, so the desired inequality holds for $o'(z) < r < k$. Continue in this manner until $i'_k + m \leq \min \{i'_{k_1}, \dots, i'_{k_j}\}$ for some j , at which point the conclusion follows.

(e) If $k = 1$, $o'(z) = 0$ so that $i'_1 + m = i_z$ and $i_z + \ell_{(z)} \geq \epsilon_1$. Now $i'_1 + m > i_{t_1}$ by assumption, hence $z < t_1$. But t_1 is the smallest integer r such that $i_r + \ell_{(r)} > \epsilon_1$; therefore $i_z + \ell_{(z)} = \epsilon_1$, by (b). Let $k > 1$ and suppose that $z \geq t_k$. Then $i_z \leq i_{t_k}$, in which case $o'(z) = o'(t_k)$. But by (c), $i'_k + m = i_z - \ell'_{(o'(z))} \leq i_{t_k} - \ell'_{(o'(t_k))}$ and by (d), $i'_k + m \leq i'_r$ for $o'(t_k) < r < k$, contradicting the hypothesis. Hence $z < t_k$. But t_k is the smallest possible integer such that $i_{t_k} + \ell_{(t_k)} - \ell_{(o(k))} > \epsilon_k$, so (e.1) follows from by (b). Results (e.2) and (e.3) are obvious.

Proof of Theorem 1:

Let $n \geq 0$. We show that if $u \otimes v$ is a component of $d^{\otimes 2} \Delta(T_{n+2})$ or $\Delta d(T_{n+2})$, there is a corresponding component that cancels it, in which case $(d^{\otimes 2} \Delta - \Delta d)(T_{n+2}) = 0$. Let $a' \otimes a$ be a component of ΔT_{n+2} . We consider various cases.

Case I. Consider a component $d^r_{(i,\ell)} \otimes 1$ of $d \otimes 1$, where $i + \ell \leq \ell'_r$, $i \geq 0$, $1 \leq \ell < \ell'_r$ and $1 \leq r \leq q + 1$. Reduce $b' = d^r_{(i,\ell)}(a')$ to an expression in second fundamental form, i.e., if $1 \leq r \leq q$, successively apply relation (1) to $d^{j+1}d^j$ for $j = q, \dots, r + 1$; apply (2) to replace $d^r_{(i,\ell)}d^r_{(i'_r,\ell'_r)}$ by $d^{r+1}_{(i'_r,\ell'_r-\ell)}d^r_{(i'_r+i,\ell)}$; then successively apply (3) to $d^{j+1}d^j$ for either $j = r - 1, \dots, \beta$ if $i + \ell < \ell'_r$ or for $j = q, \dots, \beta$ if $r = q + 1$, where β is the greatest integer such that $1 \leq \beta \leq r$ and $i + \ell \geq i'_{\beta-1}$. Then for $i + \ell < \ell'_r$, $1 \leq r \leq q + 1$ we have

$$b' = d^{q+1}_{(i'_q,\ell'_q)} \cdots d^{r+2}_{(i'_{r+1},\ell'_{r+1})} d^{r+1}_{(i'_r,\ell'_r-\ell)} d^r_{(i'_{r-1}-\ell,\ell'_{r-1})} \cdots d^{\beta+1}_{(i'_{\beta}-\ell,\ell'_{\beta})} d^{\beta}_{(i'_r+i,\ell)} d^{\beta-1}_{(i'_{\beta-1},\ell'_{\beta-1})} \cdots d^1_{(i'_1,\ell'_1)}(T_{n+2}),$$

and for $i + \ell = \ell'_r$, $1 \leq r \leq q + 1$ we have

$$b' = d^{q+1}_{(i'_q,\ell'_q)} \cdots d^{r+2}_{(i'_{r+1},\ell'_{r+1})} d^{r+1}_{(i'_r,\ell'_r-\ell)} d^r_{(i'_r+i,\ell)} d^{r-1}_{(i'_{r-1},\ell'_{r-1})} \cdots d^1_{(i'_1,\ell'_1)}(T_{n+2}).$$

Case Ia. Let $i + \ell < \ell'_r$ and $i'_r + i + \ell > i'_{\beta-1}$. As in the case of a' above, the inequalities for lower indices in an expression of b' as a face of T_{n+2} in first fundamental form are strict, but whose number now is increased by one. Obviously we will have that

$$\epsilon_{\beta} = i'_r + i + \ell'_{(\beta-1)} + \ell = i_k$$

for certain $1 \leq k \leq p$. Apply the k^{th} left-transfer to obtain

$$a = \tau_{\ell}^k(T_{n+2}) = d^{\tilde{k}}_{(\tilde{i},\tilde{\ell})}(b),$$

where $\tilde{k} = \alpha - 1$, $\tilde{i} = i_k - i_{\alpha} - \ell_{(\alpha-1)} - \ell_{(k)}$, $\tilde{\ell} = \ell_k$ and α is the smallest integer $k \leq \alpha \leq p + 1$ with $i_{\alpha} + \ell_{(\alpha)} - \ell_{(k)} \geq i_k$. Then $b' \otimes b$ is a component of ΔT_{n+2} as well.

Case Ib. Let $i + \ell < \ell'_r$ and $i'_r + i + \ell = i'_{\beta-1}$. Apply the $(\beta - 1)^{\text{th}}$ left-transfer to obtain

$$b' = \tau_{\ell}^{\beta-1}(T_{n+2}) = d^{\beta-1}_{(\tilde{i},\tilde{\ell})}(c'),$$

where $\tilde{i} = \ell$ and $\tilde{\ell} = \ell'_{\beta-1}$. Then $c' \otimes a$ is also a component of ΔT_{n+2} .

Case Ic. Let $i + \ell = \ell'_r$; there are two subcases:

Subcase i. If $1 \leq r \leq q$ and $i'_r + i \leq \min(i_{t_r} - \ell'_{(o'(t_r))}, i'_j)$, $o'(t_r) < j < r$, apply the $(r+1)^{th}$ left-transfer to obtain

$$b' = \tau_\ell^{r+1}(T_{n+2}) = d_{(\tilde{i}, \tilde{\ell})}^{\tilde{r}}(c'),$$

where $\tilde{r} = \alpha$, $\tilde{i} = i'_r - i'_\alpha$, $\tilde{\ell} = \ell'_r - \ell$ and α is the smallest integer $r \leq \alpha \leq p+1$ with $i'_\alpha \leq i'_r$. Although certain i' 's are increased by i , while ℓ'_r is reduced by i to obtain ℓ , it is straightforward to check that $c' \otimes a$ is also a component of ΔT_{n+2} .

Subcase ii. Suppose that $1 \leq r \leq q+1$ and $i'_r + i > \min(i_{t_r} - \ell'_{(o'(t_r))}, i'_j)$, $o'(t_r) < j < r$. In view of Lemma 2 (with $k = r$ and $m = i$) we have

$$\epsilon_r = i_z + \ell_{(z)} - \ell_{(o(r))} \text{ and } i'_r + i = i_z - \ell'_{(o'(z))},$$

from which we also establish the equality

$$\ell_{(z)} - \ell_{(o(r))} = \ell'_{(r-1)} + \ell - \ell'_{(o'(z))}.$$

The hypotheses of Lemma 1 are satisfied by setting $k_1 = z$, $p_1 = o(r)$ and $k_2 = r$, $p_2 = o'(z)$. Hence, $b' \otimes a$ is a component of $\Delta d_{(\tilde{i}, \tilde{\ell})}(T_{n+2})$ with $\tilde{i} = i_z$ and $\tilde{\ell} = \ell_{(z)} - \ell_{(o(r))}$.

Case II. Consider a component $1 \otimes d_{(i, \ell)}^r$ of $1 \otimes d$, where $i + \ell \leq \ell_r$, $i \geq 0$, $1 \leq \ell < \ell_r$, and $1 \leq r \leq p+1$. Reduce $b = d_{(i, \ell)}^r(a)$ to the first fundamental form, i.e., if $1 \leq r \leq p$, successively apply relation (1) to $d^{j+1}d^j$ for $j = p, \dots, r+1$, and apply (2) to replace $d_{(i, \ell)}^r d_{(i_r, \ell_r)}^r$ by $d_{(i_r, \ell_r - \ell)}^{r+1} d_{(i_r + i, \ell)}^r$. Then if $i > 0$, successively apply (3) to $d^{j+1}d^j$, for $j = r-1, \dots, \beta$; or if $r = p+1$, successively apply (3) to $d^{j+1}d^j$ for $j = p, \dots, \beta$, where β is the greatest integer with $1 \leq \beta \leq r$ and $i_r + i + \ell_{r-1} + \dots + \ell_\beta \leq i_{\beta-1}$, that is β is determined by (3.2). Then for $\beta = r$, we have $i_r + i \leq i_{r-1}$; and for $i > 0$, $1 \leq r \leq p+1$,

$$b = d_{(i_p, \ell_p)}^{p+1} \cdots d_{(i_{r+1}, \ell_{r+1})}^{r+2} d_{(i_r, \ell_r - \ell)}^{r+1} d_{(i_{r-1}, \ell_{r-1})}^r \cdots d_{(i_\beta, \ell_\beta)}^{\beta+1} d_{(\tilde{i}, \ell)}^\beta d_{(i_{\beta-1}, \ell_{\beta-1})}^{\beta-1} \cdots d_{(i_1, \ell_1)}^1(T_{n+2}),$$

where $\tilde{i} = i_r + i + \ell_{(r-1)} - \ell_{(\beta-1)}$, and

$$b = d_{(i_{p-1}, \ell_{p-1})}^p \cdots d_{(i_{r+1}, \ell_{r+1})}^{r+2} d_{(i_r, \ell_r - \ell)}^{r+1} d_{(i_r, \ell)}^r d_{(i_{r-1}, \ell_{r-1})}^{r-1} \cdots d_{(i_1, \ell_1)}^1(T_{n+2}),$$

for $i = 0$, $1 \leq r \leq p+1$.

Case IIIa. Let $i > 0$ and $i_r + i + \ell_{(r-1)} - \ell_{(\beta-1)} < i_{\beta-1}$. Once again, the inequalities for the lower indices in the expression of b in first fundamental are strict inequalities as they were for a but whose number now is increased by one. Obviously we will have that

$$i_\beta = i_r + i + \ell_{(r-1)} - \ell_{(\beta-1)} = \epsilon_k$$

for certain $1 \leq k \leq q$. Apply the k^{th} left-transfer to obtain

$$a' = \tau_\ell^k(T_{n+2}) = d_{(\tilde{i}, \tilde{\ell})}^{\tilde{k}}(b'),$$

where $\tilde{k} = \alpha - 1$, $\tilde{i} = i'_k - i'_\alpha$, $\tilde{\ell} = \ell'_k$ and α is the smallest integer $k \leq \alpha \leq q+1$ with $i'_\alpha \leq i'_k$. Then $b' \otimes b$ is a component of ΔT_{n+2} as well.

Case IIIb. Let $i > 0$ and $i_r + i + \ell_{(r-1)} - \ell_{(\beta-1)} = i_{\beta-1}$. Apply the $(\beta-1)^{th}$ left-transfer to obtain

$$b = \tau_\ell^{\beta-1}(T_{n+2}) = d_{(\tilde{i}, \tilde{\ell})}^{\beta-1}(c),$$

where $\tilde{i} = 0$ and $\tilde{\ell} = \ell_{\beta-1}$. Then $a' \otimes c$ is also a component of ΔT_{n+2} .

Case IIc. Let $i = 0$; there are two subcases:

Subcase i. If $1 \leq r \leq p$ and no integer $1 \leq k \leq q$ exists with $r = t_k$, then

$$\epsilon_k = i_r + \ell + \ell_{(r-1)} - \ell_{(o(k))} \quad \text{and} \quad i'_k = i_r - \ell'_{(o'(r))}.$$

Apply the $(r+1)^{th}$ left-transfer to obtain

$$b = \tau_\ell^{r+1}(T_{n+2}) = d_{(\tilde{i}, \tilde{\ell})}^{\tilde{r}}(c),$$

where $\tilde{r} = \alpha$, $\tilde{i} = i_r - i_\alpha + \ell_{(\alpha-1)} - \ell_{(r)}$, $\tilde{\ell} = \ell_r - \ell$ and α is the smallest integer $r < \alpha \leq p+1$ such that $i_\alpha + \ell_{(\alpha)} - \ell_{(r)} \geq i_r$; namely, $\alpha = r+1$ or $\alpha = t_{o'(r)}$. Now the required inequality for the i'_k 's could conceivably be violated for $k > o'(r)$, but this is not so since it is easy to see that: (a) if r is not realized as t_k for some $k > o'(r)$, then each z with $\alpha < z < r$ is so; and (b) if $r = t_k$ for some $k > o'(r)$, while $\epsilon_k = i_r + \ell + \ell_{(r-1)} - \ell_{(o(k))}$, then α (and not r) serves as t_k for indices of face operators in expressions of a and c ; moreover, for either $\alpha = r+1$ or $\alpha = t_{o'(r)}$ (in which case $i'_k \leq i'_{o'(r)}$), one has $i'_k \leq \min_{o'(\alpha) < j < k} \{i'_j, i_\alpha - \ell'_{(o'(\alpha))}\}$ so that $a' \otimes c$ is also a component of ΔT_{n+2} .

Subcase ii. If $1 \leq r \leq p+1$ and $r = t_k$ for some $1 \leq k \leq q$, then

$$\epsilon_k = i_r + \ell + \ell_{(r-1)} - \ell_{(o(k))} \quad \text{and} \quad i'_k = i_r - \ell'_{(o'(r))},$$

from which we establish the equality

$$\ell + \ell_{(r-1)} - \ell_{(o(k))} = \ell'_{(k)} - \ell'_{(o(r))}.$$

The hypotheses of Lemma 1 are satisfied by putting $k_1 = r = t_k$, $p_1 = o(k)$ and $k_2 = k$, $p_2 = o'(r)$ so that $a' \otimes c$ is a component of $\Delta d_{(\tilde{i}, \tilde{\ell})}(T_{n+2})$ with $\tilde{i} = i_r$ and $\tilde{\ell} = \ell + \ell_{(r-1)} - \ell_{(o(k))}$.

Case III. Let $c' \otimes c$ be a component of $\Delta d_{(i, \ell)}(T_{n+2})$. Reduce c and c' to the first and second fundamental forms, respectively. According to Lemma 1, we have either $i = i_{r_1}$ or $i = i'_{r_2} + \ell'_{r_2} + \dots + \ell_{k_2} = i'_{r_2+1} + \ell'_{r_2+1}$ for certain integers r_1, r_2, k_2 , i.e., $i_{r_1+1} = i_{r_1}$ or $i'_{r_2+1} + \ell'_{r_2+1} = i'_{r_2}$. Note that the shuffles under consideration prevent both cases from occurring simultaneously, and we obtain the situation dual to either Subcase ii of Case IIc, or to Subcase ii of Case Ic. Thus we obtain components $c' \otimes a$ or $a' \otimes c$ of ΔT_{n+2} with $d_{(i, \ell)}^{r_1}(a) = c$ or $d_{(i, \ell)}^{r_2}(a') = c'$, respectively.

5. APPLICATION: TENSOR PRODUCTS OF A_∞ -(CO)ALGEBRAS

In this section, we use the diagonal Δ to define the tensor product of A_∞ -(co)algebras in maximal generality. We note that a special case was given by J. Smith [8] for certain objects with a richer structure than we have here. We also mention that Lada and Markl [3] defined an A_∞ tensor product structure on a construct different from the tensor product of graded modules.

We adopt the following notation and conventions: Let R be a commutative ring with unity; R -modules are assumed to be \mathbb{Z} -graded, tensor products and Hom 's are defined over R and all maps are R -module maps unless otherwise indicated. If an R -module V is connected, $\overline{V} = V/V_0$. The symbol $1 : V \rightarrow V$ denotes the identity map; the suspension and desuspension maps \uparrow and \downarrow shift dimension by $+1$ and -1 , respectively. Define $V^{\otimes 0} = R$ and $V^{\otimes n} = V \otimes \dots \otimes V$ with $n > 0$ factors; then $TV = \bigoplus_{n \geq 0} V^{\otimes n}$ and $T^a V$ (respectively, $T^c V$) denotes the free tensor algebra (respectively, cofree tensor coalgebra) of V . Given R -modules

V_1, \dots, V_n , a permutation $\sigma \in S_n$ induces an isomorphism $\sigma : V_1 \otimes \dots \otimes V_n \rightarrow V_{\sigma^{-1}(1)} \otimes \dots \otimes V_{\sigma^{-1}(n)}$ by $\sigma(x_1 \cdots x_n) = \pm x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}$, where \pm is the Koszul sign. In particular, $\sigma_{2,n} = (1 \ 3 \ \cdots \ (2n-1) \ 2 \ 4 \ \cdots \ 2n) : (A \otimes B)^{\otimes n} \rightarrow A^{\otimes n} \otimes B^{\otimes n}$ and $\sigma_{n,2} = \sigma_{2,n}^{-1}$ induce isomorphisms $(\sigma_{2,n})^* : \text{Hom}(A^{\otimes n} \otimes B^{\otimes n}, A \otimes B) \rightarrow \text{Hom}((A \otimes B)^{\otimes n}, A \otimes B)$ and $(\sigma_{n,2})_* : \text{Hom}(A \otimes B, A^{\otimes n} \otimes B^{\otimes n}) \rightarrow \text{Hom}(A \otimes B, (A \otimes B)^{\otimes n})$. The map $\iota : \text{Hom}(U, V) \otimes \text{Hom}(U', V') \rightarrow \text{Hom}(U \otimes U', V \otimes V')$ is the canonical isomorphism. If $f : V^{\otimes p} \rightarrow V^{\otimes q}$ is a map, we let $f_{i,n-p-i} = 1^{\otimes i} \otimes f \otimes 1^{\otimes n-p-i} : V^{\otimes n} \rightarrow V^{\otimes n-p+q}$, where $0 \leq i \leq n-p$. The abbreviations *DGM*, *DGA*, and *DGC* stand for *differential graded R-module*, *DG R-algebra* and *DG R-coalgebra*, respectively.

We begin with a review of A_∞ -(co)algebras paying particular attention to the signs. Let A be a connected R -module equipped with operations $\{\varphi^k \in \text{Hom}^{k-2}(A^{\otimes k}, A)\}_{k \geq 1}$. For each k and $n \geq 1$, linearly extend φ^k to $A^{\otimes n}$ via

$$\sum_{i=0}^{n-k} \varphi_{i,n-k-i}^k : A^{\otimes n} \rightarrow A^{\otimes n-k+1},$$

and consider the induced map of degree -1 given by

$$\sum_{i=0}^{n-k} (\uparrow \varphi^k \downarrow^{\otimes k})_{i,n-k-i} : (\uparrow \overline{A})^{\otimes n} \rightarrow (\uparrow \overline{A})^{\otimes n-k+1}.$$

Let $\tilde{B}A = T^c(\uparrow \overline{A})$ and define a map $d_{\tilde{B}A} : \tilde{B}A \rightarrow \tilde{B}A$ of degree -1 by

$$(5.1) \quad d_{\tilde{B}A} = \sum_{\substack{1 \leq k \leq n \\ 0 \leq i \leq n-k}} (\uparrow \varphi^k \downarrow^{\otimes k})_{i,n-k-i}.$$

The identities $(-1)^{[n/2]} \uparrow^{\otimes n} \downarrow^{\otimes n} = 1^{\otimes n}$ and $[n/2] + [(n+k)/2] \equiv nk + [k/2] \pmod{2}$ imply that

$$(5.2) \quad d_{\tilde{B}A} = \sum_{\substack{1 \leq k \leq n \\ 0 \leq i \leq n-k}} (-1)^{[(n-k)/2] + i(k+1)} \uparrow^{\otimes n-k+1} \varphi_{i,n-k-i}^k \downarrow^{\otimes n}.$$

Definition 7. $(A, \varphi^n)_{n \geq 1}$ is an A_∞ -algebra if $d_{\tilde{B}A}^2 = 0$.

Proposition 2. For each $n \geq 1$, the operations $\{\varphi^n\}$ on an A_∞ -algebra satisfy the following quadratic relations:

$$(5.3) \quad \sum_{\substack{0 \leq \ell \leq n-1 \\ 0 \leq i \leq n-\ell-1}} (-1)^{\ell(i+1)} \varphi^{n-\ell} \varphi_{i,n-\ell-1-i}^{\ell+1} = 0.$$

Proof. For $n \geq 1$,

$$\begin{aligned}
0 &= \sum_{\substack{1 \leq k \leq n \\ 0 \leq i \leq n-k}} (-1)^{[(n-k)/2]+i(k+1)} \uparrow \varphi^{n-k+1} \downarrow^{\otimes n-k+1} \uparrow^{\otimes n-k+1} \varphi_{i,n-k-i}^k \downarrow^{\otimes n} \\
&= \sum_{\substack{1 \leq k \leq n \\ 0 \leq i \leq n-k}} (-1)^{n-k+i(k+1)} \varphi^{n-k-1} \varphi_{i,n-k-i}^k \\
&= -(-1)^n \sum_{\substack{0 \leq \ell \leq n-1 \\ 0 \leq i \leq n-\ell-1}} (-1)^{\ell(i+1)} \varphi^{n-\ell} \varphi_{i,n-\ell-1-i}^{\ell+1}.
\end{aligned}$$

□

It is easy to prove that

Proposition 3. *If $(A, \varphi^n)_{n \geq 1}$ is an A_∞ -algebra, then $(\tilde{B}A, d_{\tilde{B}A})$ is a DGC.*

Definition 8. *Let $(A, \varphi^n)_{n \geq 1}$ be an A_∞ -algebra. The tilde bar construction on A is the DGC $(\tilde{B}A, d_{\tilde{B}A})$.*

Definition 9. *Let A and C be A_∞ -algebras. A chain map $f = f^1 : A \rightarrow C$ is a map of A_∞ -algebras if there exists a sequence of maps $\{f^k \in \text{Hom}^{k-1}(A^{\otimes k}, C)\}_{k \geq 2}$ such that*

$$\tilde{f} = \sum_{n \geq 1} \left(\sum_{k \geq 1} \uparrow f^k \downarrow^{\otimes k} \right)^{\otimes n} : \tilde{B}A \rightarrow \tilde{B}C$$

is a DGC map.

Dually, consider a sequence of operations $\{\psi^k \in \text{Hom}^{k-2}(A, A^{\otimes k})\}_{k \geq 1}$. For each k and $n \geq 1$, linearly extend each ψ^k to $A^{\otimes n}$ via

$$\sum_{i=0}^{n-1} \psi_{i,n-1-i}^k : A^{\otimes n} \rightarrow A^{\otimes n+k-1},$$

and consider the induced map of degree -1 given by

$$\sum_{i=0}^{n-1} (\downarrow^{\otimes k} \psi^k \uparrow)_{i,n-1-i} : (\downarrow \overline{A})^{\otimes n} \rightarrow (\downarrow \overline{A})^{\otimes n+k-1}.$$

Let $\tilde{\Omega}A = T^a(\downarrow \overline{A})$ and define a map $d_{\tilde{\Omega}A} : \tilde{\Omega}A \rightarrow \tilde{\Omega}A$ of degree -1 by

$$d_{\tilde{\Omega}A} = \sum_{\substack{n,k \geq 1 \\ 0 \leq i \leq n-1}} (\downarrow^{\otimes k} \psi^k \uparrow)_{i,n-1-i},$$

which can be rewritten as

$$(5.4) \quad d_{\tilde{\Omega}A} = \sum_{\substack{n,k \geq 1 \\ 0 \leq i \leq n-1}} (-1)^{[n/2]+i(k+1)+k(n+1)} \downarrow^{\otimes n+k-1} \psi_{i,n-1-i}^k \uparrow^{\otimes n}.$$

Definition 10. *$(A, \psi^n)_{n \geq 1}$ is an A_∞ -coalgebra if $d_{\tilde{\Omega}A}^2 = 0$.*

Proposition 4. *For each $n \geq 1$, the operations $\{\psi^k\}$ on an A_∞ -coalgebra satisfy the following quadratic relations:*

$$(5.5) \quad \sum_{\substack{0 \leq \ell \leq n-1 \\ 0 \leq i \leq n-\ell-1}} (-1)^{\ell(n+i+1)} \psi_{i, n-\ell-1-i}^{\ell+1} \psi^{n-\ell} = 0.$$

Proof. The proof is similar to the proof of Proposition 2 and is omitted. \square

Again, it is easy to prove that

Proposition 5. *If $(A, \psi^n)_{n \geq 1}$ is an A_∞ -coalgebra, then $(\tilde{\Omega}A, d_{\tilde{\Omega}A})$ is a DGA.*

Definition 11. *Let $(A, \psi^n)_{n \geq 1}$ be an A_∞ -coalgebra. The tilde cobar construction on A is the DGA $(\tilde{\Omega}A, d_{\tilde{\Omega}A})$.*

Definition 12. *Let A and B be A_∞ -coalgebras. A chain map $g = g^1 : A \rightarrow B$ is a map of A_∞ -coalgebras if there exists a sequence of maps $\{g^k \in \text{Hom}^{k-1}(A, B^{\otimes k})\}_{k \geq 2}$ such that*

$$(5.6) \quad \tilde{g} = \sum_{n \geq 1} \left(\sum_{k \geq 1} \downarrow^{\otimes k} g^k \uparrow \right)^{\otimes n} : \tilde{\Omega}A \rightarrow \tilde{\Omega}B,$$

is a DGA map.

The structure of an A_∞ -(co)algebra is encoded by the quadratic relations among its operations (also called “higher homotopies”). Although the “direction,” i.e., sign, of these higher homotopies is arbitrary, each choice of directions determines a set of signs in the quadratic relations, the “simplest” of which appears on the algebra side when no changes of direction are made; see (5.1) and (5.3) above. Interestingly, the “simplest” set of signs appear on the coalgebra side when ψ^n is replaced by $(-1)^{\lfloor (n-1)/2 \rfloor} \psi^n$, $n \geq 1$, i.e., the direction of every third and fourth homotopy is reversed. The choices one makes will depend on the application; for us the appropriate choices are as in (5.3) and (5.5).

Let $\mathcal{A}_\infty = \oplus_{n \geq 2} C_*(K_n)$ and let $(A, \varphi^n)_{n \geq 1}$ be an A_∞ -algebra with quadratic relations as in (5.3). For each $n \geq 2$, associate $e^{n-2} \in C_{n-2}(K_n)$ with the operation φ^n via

$$(5.7) \quad e^{n-2} \mapsto (-1)^n \varphi^n$$

and each codimension 1 face $d_{(i, \ell)}(e^{n-2}) \in C_{n-3}(K_n)$ with the quadratic composition

$$(5.8) \quad d_{(i, \ell)}(e^{n-2}) \mapsto \varphi^{n-\ell} \varphi_{i, n-\ell-1-i}^{\ell+1}.$$

Then (5.7) and (5.8) induce a chain map

$$(5.9) \quad \zeta_A : \mathcal{A}_\infty \longrightarrow \oplus_{n \geq 2} \text{Hom}^*(A^{\otimes n}, A)$$

representing the A_∞ -algebra structure on A . Dually, if $(A, \psi^n)_{n \geq 1}$ is an A_∞ -coalgebra with quadratic relations as in (5.5), the associations

$$e^{n-2} \mapsto \psi^n \text{ and } d_{(i, \ell)}(e^{n-2}) \mapsto \psi_{i, n-\ell-1-i}^{\ell+1} \psi^{n-\ell}$$

induce a chain map

$$(5.10) \quad \xi_A : \mathcal{A}_\infty \longrightarrow \oplus_{n \geq 2} \text{Hom}^*(A, A^{\otimes n})$$

representing the A_∞ -coalgebra structure on A . The definition of the tensor product is now immediate:

Definition 13. *The tensor product of A_∞ -algebras (A, ζ_A) and (B, ζ_B) is given by*

$$(A, \zeta_A) \otimes (B, \zeta_B) = (A \otimes B, \zeta_{A \otimes B}),$$

where $\zeta_{A \otimes B}$ is the sum of the compositions

$$\begin{array}{ccc} C_*(K_n) & \xrightarrow{\zeta_{A \otimes B}} & \text{Hom}((A \otimes B)^{\otimes n}, A \otimes B) \\ \Delta_K \downarrow & & \uparrow (\sigma_{2,n})^* \iota \end{array}$$

$$C_*(K_n) \otimes C_*(K_n) \xrightarrow{\zeta_A \otimes \zeta_B} \text{Hom}(A^{\otimes n}, A) \otimes \text{Hom}(B^{\otimes n}, B)$$

over all $n \geq 2$; the A_∞ -algebra operations Φ^n on $A \otimes B$ are given by

$$\Phi^n = (\sigma_{2,n})^* \iota (\zeta_A \otimes \zeta_B) \Delta_K (e^{n-2}).$$

Dually, the tensor product of A_∞ -coalgebras (A, ξ_A) and (B, ξ_B) is given by

$$(A, \xi_A) \otimes (B, \xi_B) = (A \otimes B, \xi_{A \otimes B}),$$

where $\xi_{A \otimes B}$ is the sum of the compositions

$$\begin{array}{ccc} C_*(K_n) & \xrightarrow{\xi_{A \otimes B}} & \text{Hom}(A \otimes B, (A \otimes B)^{\otimes n}) \\ \Delta_K \downarrow & & \uparrow (\sigma_{n,2})_* \iota \end{array}$$

$$C_*(K_n) \otimes C_*(K_n) \xrightarrow{\xi_A \otimes \xi_B} \text{Hom}(A, A^{\otimes n}) \otimes \text{Hom}(B, B^{\otimes n})$$

over all $n \geq 2$; the A_∞ -coalgebra operations Ψ^n on $A \otimes B$ are given by

$$\Psi^n = (\sigma_{n,2})_* \iota (\xi_A \otimes \xi_B) \Delta_K (e^{n-2}).$$

Example 3. *If $(A, \psi^n)_{n \geq 1}$ is an A_∞ -coalgebra, the A_∞ operations on $A \otimes A$ are:*

$$\Psi^1 = \psi^1 \otimes 1 + 1 \otimes \psi^1$$

$$\Psi^2 = \sigma_{2,2} (\psi^2 \otimes \psi^2)$$

$$\Psi^3 = \sigma_{3,2} (\psi_0^2 \psi_0^2 \otimes \psi^3 + \psi^3 \otimes \psi_1^2 \psi_0^2)$$

$$\begin{aligned} \Psi^4 = \sigma_{4,2} & (\psi_0^2 \psi_0^2 \psi_0^2 \otimes \psi^4 + \psi^4 \otimes \psi_2^2 \psi_1^2 \psi_0^2 + \psi_0^3 \psi_0^2 \otimes \psi_1^2 \psi_0^3 \\ & + \psi_0^3 \psi_0^2 \otimes \psi_1^3 \psi_0^2 + \psi_1^2 \psi_0^3 \otimes \psi_1^3 \psi_0^2 - \psi_0^2 \psi_0^3 \otimes \psi_2^2 \psi_0^3) \end{aligned}$$

$$\begin{aligned} \Psi^5 = \sigma_{5,2} & (\psi_0^2 \psi_0^2 \psi_0^2 \psi_0^2 \otimes \psi^5 + \psi^5 \otimes \psi_3^2 \psi_2^2 \psi_1^2 \psi_0^2 \\ & + \psi_0^4 \psi_0^2 \otimes \psi_2^2 \psi_1^2 \psi_0^2 - \psi_1^2 \psi_0^4 \otimes \psi_3^2 \psi_1^3 \psi_0^2 + \psi_0^3 \psi_0^2 \psi_0^2 \otimes \psi_1^3 \psi_0^2 \\ & + \psi_0^4 \psi_0^2 \otimes \psi_2^2 \psi_1^3 \psi_0^2 - \psi_1^3 \psi_0^3 \otimes \psi_2^2 \psi_1^3 \psi_0^2 + \psi_0^3 \psi_0^2 \psi_0^2 \otimes \psi_1^4 \psi_0^2 \\ & + \psi_0^4 \psi_0^2 \otimes \psi_2^2 \psi_1^2 \psi_0^3 - \psi_1^3 \psi_0^3 \otimes \psi_2^2 \psi_1^2 \psi_0^2 - \psi_0^2 \psi_0^3 \psi_0^2 \otimes \psi_2^2 \psi_0^4 \\ & + \psi_0^2 \psi_0^4 \otimes \psi_3^2 \psi_2^2 \psi_0^3 + \psi_0^2 \psi_0^2 \psi_0^3 \otimes \psi_3^2 \psi_0^4 + \psi_0^2 \psi_0^3 \psi_0^2 \otimes \psi_3^2 \psi_0^3 \\ & + \psi_2^2 \psi_0^4 \otimes \psi_3^2 \psi_1^2 \psi_0^2 + \psi_0^2 \psi_1^2 \psi_0^3 \otimes \psi_2^2 \psi_0^3 + \psi_1^2 \psi_0^3 \psi_0^2 \otimes \psi_3^2 \psi_0^3 \\ & - \psi_0^3 \psi_0^3 \otimes \psi_2^2 \psi_1^2 \psi_0^3 + \psi_1^2 \psi_1^2 \psi_0^3 \otimes \psi_4^2 \psi_0^2 + \psi_1^2 \psi_0^3 \psi_0^2 \otimes \psi_4^2 \psi_0^2 \\ & - \psi_0^3 \psi_0^3 \otimes \psi_3^2 \psi_1^3 \psi_0^2 + \psi_0^3 \psi_0^2 \psi_0^2 \otimes \psi_1^2 \psi_0^4), \end{aligned}$$

etc.

Note that the compositions in Definition 13 only use the operations ψ^n and not the quadratic relations (5.5). Indeed, one can iterate an arbitrary family of operations $\{\psi^n\}$ as in Example (3) to produce iterated operations $\Psi^n : A^{\otimes k} \rightarrow (A^{\otimes k})^{\otimes n}$ whether or not (A, ψ^n) is an A_∞ -coalgebra. Of course, the Ψ^n 's define an A_∞ -coalgebra structure on $A^{\otimes k}$ whenever $d_{\Omega(A^{\otimes k})}^2 = 0$, and we make extensive use of this fact in the sequel [7]. Finally, since Δ_K is homotopy coassociative (not strict), the tensor product only iterates up to homotopy. In the sequel we always coassociate on the extreme left.

6. APPENDIX: ASSOCIAHEDRAL SETS

An associahedral set is a combinatorial object generated by Stasheff associahedra K and equipped with appropriate face and degeneracy operators. Associahedral sets are similar in many ways to simplicial or cubical sets.

6.1. Singular associahedral sets. To motivate the notion of an associahedral set, we begin with a construction of singular associahedral sets, our universal example. Let X be a topological space. Define the singular associahedral complex $Sing^K X$ as follows: Let

$$(Sing^K X)_{n-k+2}^{(j_1, n_1), \dots, (j_{k+1}, n_{k+1})} = \{\text{Continuous maps } K_{n_1+2} \times \dots \times K_{n_{k+1}+2} \rightarrow X\},$$

where $\sum_{q=1}^{k+1} n_q = n - k$, $n_q \geq 0$, $0 \leq k \leq n$, $n \geq j_1 \geq \dots \geq j_k \geq j_{k+1} = 0$ and $K_{n_1+2} \times \dots \times K_{n_{k+1}+2}$ is a Cartesian product of associahedra. Let

$$\begin{aligned} \delta^q_{(i_q, \ell_q)} : K_{n_1+2} \times \dots \times K_{n_{\beta-1}+2} \times K_{\ell_q+1} \times \dots \\ \dots \times K_{n_q-\ell_q+2} \times K_{n_{q+1}+2} \times \dots \times K_{n_{k+2}+2} \\ \rightarrow K_{n_1+2} \times \dots \times K_{n_q+2} \times \dots \times K_{n_{k+2}+2}, \end{aligned}$$

be the map determined by $\delta^q_{(i_q, \ell_q)} = \delta'^q_{(i_q, \ell_q)} \circ T_{(q, \beta)}$, where

$$\begin{aligned} \delta'^q_{(i_q, \ell_q)} : K_{n_1+2} \times \dots \times K_{\ell_q+1} \times K_{n_q-\ell_q+2} \times \dots \times K_{n_{k+2}+2} \\ \rightarrow K_{n_1+2} \times \dots \times K_{n_q+2} \times \dots \times K_{n_{k+2}+2}, \end{aligned}$$

$$\delta^q_{(i_q, \ell_q)} = 1^{\times q-1} \times \delta''_{(i_q, \ell_q)} \times 1^{\times k+1-q},$$

$$\delta''_{(i_q, \ell_q)} : K_{\ell_q+1} \times K_{n_q-\ell_q+2} \rightarrow K_{n_q+2}$$

is the standard inclusion corresponding to the pair (i_q, ℓ_q) , and

$$\begin{aligned} T_{(q, \beta)} : K_{n_1+2} \times \dots \times K_{n_{\beta-1}+2} \times K_{\ell_q+1} \times \dots \\ \dots \times K_{n_q-\ell_q+2} \times K_{n_{q+1}+2} \times \dots \times K_{n_{k+2}+2} \\ \xrightarrow{\cong} K_{n_1+2} \times \dots \times K_{n_{\beta-1}+2} \times \dots \\ \dots \times K_{\ell_q+1} \times K_{n_q-\ell_q+2} \times K_{n_{q+1}+2} \times \dots \times K_{n_{k+2}+2} \end{aligned}$$

is the permutation isomorphism in which β is defined by (3.2). Let

$$\eta_i^q : K_{n_1+2} \times \dots \times K_{n_q+3} \times \dots \times K_{n_{k+1}+2} \rightarrow K_{n_1+2} \times \dots \times K_{n_q+2} \times \dots \times K_{n_{k+1}+2}$$

be the projection (cf. [9]). Then for $f \in (Sing^K X)_{n-k+2}^{(j_1, n_1), \dots, (j_{k+1}, n_{k+1})}$, define

$$\begin{aligned} d^q_{(i_q, \ell_q)} : (Sing^K X)_{n-k+2}^{(j_1, n_1), \dots, (j_{k+1}, n_{k+1})} \rightarrow \\ (Sing^K X)_{n-k+1}^{(j_1, n_1), \dots, (j_{\beta-1}, n_{\beta-1}), (j(q, \beta), \ell_q-1), \dots, (j_q, n_q-\ell_q), \dots, (j_{k+1}, n_{k+1})}, \end{aligned}$$

with $j(q, \beta)$ is defined in (3.1) and

$$s_i^q : (Sing^K X)_{n-k+2}^{(j_1, n_1), \dots, (j_{k+1}, n_{k+1})} \rightarrow (Sing^K X)_{n-k+3}^{(j_1, n_1), \dots, (j_q, n_q+1), \dots, (j_{k+1}, n_{k+1})},$$

as compositions

$$d_{(i_q, \ell_q)}^q(f) = f \circ \delta_{(i_q, \ell_q)}^q \text{ and } s_i^q(f) = \eta_i^q \circ f.$$

Given the abstract definition below, one can easily check that $(Sing^K X, d_{(i_q, \ell_q)}^q, s_i^q)$ is an associahedral set.

6.2. Abstract associahedral sets.

Definition 14. *An associahedral set is a graded set*

$$\mathcal{K} = \{K_{n-k+2}^{(j_1, n_1), \dots, (j_{k+1}, n_{k+1})} \mid n \geq j_1 \geq \dots \geq j_{k+1} = 0, n_q \geq 0, n_{(k+1)} = n - k\}_{0 \leq k \leq n},$$

together with face and degeneracy operators defined for $1 \leq q \leq k+1$:

$$d_{(i_q, \ell_q)}^q : K_{n-k+2}^{(j_1, n_1), \dots, (j_{k+1}, n_{k+1})} \rightarrow K_{n-k+1}^{(j_1, n_1), \dots, (j_{\beta-1}, n_{\beta-1}), (j(q, \beta), \ell_q-1), (j_\beta, n_\beta), \dots, (j_{q-1}, n_{q-1}), (j_q, n_q-\ell_q), (j_{q+1}, n_{q+1}), \dots, (j_{k+1}, n_{k+1})}$$

where $j(q, \beta)$ and β are defined in (3.1) and (3.2), $0 \leq i_q \leq n_q$; $1 \leq \ell_q \leq n_q$; $i_q + \ell_q \leq n_q + 1$, and

$$s_j^q : K_{n-k+2}^{(j_1, n_1), \dots, (j_{k+1}, n_{k+1})} \rightarrow K_{n-k+3}^{(j_1, n_1), \dots, (j_{q-1}, n_{q-1}), (j_q, n_q+1), (j_{q+1}, n_{q+1}), \dots, (j_{k+1}, n_{k+1})}$$

for $1 \leq j \leq n_q + 3$, satisfying relations (1)-(3) as well as

$$\begin{aligned} d_{(i, \ell)}^p s_j^q &= s_j^{q+1} d_{(i, \ell)}^p; & p < q \\ d_{(i, \ell)}^p s_j^q &= s_j^q d_{(i, \ell)}^p; & p > q \\ d_{(i, \ell)}^q s_j^q &= s_{j-\ell}^{q+1} d_{(i, \ell)}^q; & i + \ell + 1 < j \\ d_{(i, \ell)}^q s_j^q &= s_{j-i}^q d_{(i, \ell-1)}^q; & i < j < i + \ell + 2, \ell > 1 \\ d_{(i, \ell)}^q s_j^q &= s_j^{q+1} d_{(i-1, \ell)}^q; & i \geq j, \ell \leq n_q \\ d_{(i, \ell)}^q s_j^q &= 1; & (i, \ell) = (j-1, 1), 1 \leq j < n_q + 3 \\ d_{(i, \ell)}^q s_j^q &= 1; & (i, \ell) = (j-2, 1), 1 < j \leq n_q + 3 \\ d_{(i, \ell)}^q s_j^q &= 1; & (i, \ell) = (0, n_q + 1), j = n_q + 3 \\ d_{(i, \ell)}^q s_j^q &= 1; & (i, \ell) = (1, n_q + 1), j = 1 \\ s_j^p s_{j'}^q &= s_{j'}^q s_j^p; & p \neq q \\ s_j^q s_{j'}^q &= s_{j'+1}^q s_j^q; & p = q, j \leq j'. \end{aligned}$$

Given an associahedral set \mathcal{K} , let

$$(C_*(\mathcal{K}), d) = \bigoplus C_{n-k}(K_{n-k+2}^{(j_1, n_1), \dots, (j_{k+1}, n_{k+1})}, d^{n_1, \dots, n_{k+1}}),$$

where

$$d^{n_1, \dots, n_{k+1}} = \sum_{(i_q, \ell_q)} (-1)^{\epsilon_1 + \epsilon_2} d_{(i_q, \ell_q)}^q,$$

with ϵ_i defined in (3.3); define the diagonal

$$\Delta_{\mathcal{K}} : C_*(\mathcal{K}) \rightarrow C_*(\mathcal{K}) \otimes C_*(\mathcal{K})$$

on $K^{(0,n)}$ by (3.4) and extend to other components of $K^{(j_1; n_1), \dots, (j_{k+1}; n_{k+1})}$ by the formal multiplicative rule with respect to indices (n_1, \dots, n_{k+1}) , i.e., by the same formulas as on a product cell $K_{n_1+2} \times \dots \times K_{n_{k+1}+2}$. Finally, set $C_*^N(\mathcal{K}) = C_*(\mathcal{K})/D$, where D is the submodule generated by the degeneracies; then $(C_*^N(\mathcal{K}), d)$ is a chain complex equipped with a diagonal $\Delta_{\mathcal{K}}$ induced by Δ .

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A. RAZMADZE MATHEMATICAL INSTITUTE, GEORGIAN ACADEMY OF SCIENCES, M. ALEKSIDZE ST., 1, 0193 TBILISI, GEORGIA

E-mail address: SANE@rmi.acnet.ge <mailto:SANE@rmi.acnet.ge>

DEPARTMENT OF MATHEMATICS, MILLERSVILLE UNIVERSITY OF PENNSYLVANIA, MILLERSVILLE, PA. 17551

E-mail address: ron.umble@millersville.edu <mailto:ron.umble@millersville.edu>